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On the irrationality of factorial series II

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ABSTRACT

In this paper we give irrationality results for numbers of the form $\sum_{n=1}^{\infty} \frac{a_n}{n!}$ where the numbers a_n behave like a geometric progression for a while. The method is elementary, not using differentiation or integration. In particular, we derive elementary proofs of the irrationality of π and e^m for Gaussian integers $m \neq 0$.

1. Introduction

This paper deals with irrationality proofs which imply the irrationality of the numbers e^m for integer m and of π. For the history of the irrationality and transcendence of these numbers we refer to Koksma [10, p. 53] and Shidlovskij [15, pp. 77–78], for short proofs of the irrationality of the numbers to Niven [12, Ch. 2] and Nesterenko [11]. Far-reaching generalizations of these results, dealing with the transcendence and algebraically independence of values of hypergeometric functions, E-functions and G-functions, have been obtained by Shidlovskij and others. See the books of Shidlovskij [15] and Fel'dman and Nesterenko [5] and the papers mentioned in these books.

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Let $(a_n)_{n=1}^{\infty}$ be a sequence of integers. In the present paper we study the irrationality of $S := \sum_{n=1}^{\infty} \frac{a_n}{n!}$ and, more generally, of $S^* := \sum_{N=1}^{\infty} \frac{a_N}{\prod_{n=1}^{N}(an+b)}$ where a and b are given positive integers. Erdős and Straus [4] started a series of results in which the size of the difference $a_{n+1} - a_n$ is a relevant factor. They used such results to establish the irrationality of S in case $(a_n)_{n=1}^{\infty}$ represents a multiplicative or other arithmetic function. Their result, with refinements by the authors [7] and Tijdeman and Yuan [16], is that if $a_{n+1} - a_n = o(n)$ and $\frac{a_n}{n-1}$ is not ultimately constant, then S is irrational. This result generalizes Erdős' result [2] that $\sum_{n=1}^{\infty} \frac{p_n}{n!} \notin \mathbb{Q}$, where $\{p_n\}_{n=1}^{\infty}$ is the sequence of consecutive prime numbers. Erdős's claim that $\sum_{n=1}^{\infty} \frac{p_n^k}{n!} \notin \mathbb{Q}$ is irrational for $k = 2, 3, \ldots$ was recently confirmed by Schlage-Puchta [14]. By the above mentioned method Erdős and Straus [4] proved that the numbers 1, $\sum_{n=1}^{\infty} \frac{\sigma(n)}{n!}$, $\sum_{n=1}^{\infty} \frac{\phi_n}{n!}$ and $\sum_{n=1}^{\infty} \frac{a_n}{n!}$, where $|a_n| < n^{\frac{1}{2}-\epsilon}$ for all large n and $a_n \neq 0$ infinitely often, are linearly independent over the rationals. It is conjectured that $\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n!}$, where $\sigma_k(n)$ denotes the sum of the k-th powers of n, for $k = 0, 1, 2, 3, \ldots$ is irrational. This has been proved for $k = 0, 1$ by Erdős and Straus [4], for $k = 2$ by Erdős and Kac [3], and more recently, independently of each other, for $k = 3$ by Schlage-Puchta [13] and Friedlander, Luca and Stoiciu [6] independently of each other, and under some twin prime condition for $k > 3$ by the same authors.

Tijdeman and Yuan [16] started to use second order differences (cf. the proof of their Theorem 4.3). Later the authors [8,9] pursued this idea by studying K-th order differences. This enabled them to derive a variety of results in cases where the integer sequence a_n has polynomial growth.

It is the purpose of the present paper to investigate the situation that the sequence (a_n) has exponential or factorial growth. To achieve this, we have to impose a stronger regularity condition. We prove two propositions from which we derive a number of theorems. As examples of the obtained results we mention that we prove the irrationality of the numbers

$$\sum_{n=1}^{\infty} \frac{2^{\pi(n)}}{n!}, \quad \sum_{n=1}^{\infty} \frac{(\pi(n))^2}{n!}, \quad \sum_{n=1}^{\infty} \frac{2^{n\lfloor \frac{\log n}{5} \rfloor}}{n!}.$$

The second number shows that our approach also yields some new results in cases where the numerator has polynomial growth.

The method we use is based on the summation formula stated in Lemma 2.3. Thus we get elementary proofs (i.e. not using differentiation or integration) of the irrationality of e^m for integer m and for π. We note that our approach makes it possible to derive irrationality measures for the numbers for which we prove irrationality.

2. Lemmas

We study the irrationality of sums

$$S := \sum_{n=1}^{\infty} \frac{a_n}{(a+b)_{a,n}}$$

where $(x)_{a,n} = x(x+n)\cdots(x+(n-1)a)$. The following lemma dealing with the sum

$$S_N := (a+b)_{a,N-1} \sum_{n=N}^{\infty} \frac{a_n}{(a+b)_{a,n}} \tag{1}$$

is crucial. We denote by $\mathbb{G}$ the set of Gaussian integers.

Lemma 2.1. *If $S = \frac{t}{q}$ for some $t \in \mathbb{Z}$, $q \in \mathbb{N}$, then $qS_N \in \mathbb{Z}$ for all N. Similarly, if $S = \frac{t}{q}$ for some $t \in \mathbb{G}$, $q \in \mathbb{N}$, then $qS_N \in \mathbb{G}$ for all N.*

Proof. We have

$$t(a+b)_{a,N-1} = q(a+b)_{a,N-1}S = q\sum_{n=1}^{N-1} a_n \frac{(a+b)_{a,N-1}}{(a+b)_{a,n}} + qS_N \tag{2}$$

and the last term is a (Gaussian) integer, since all the others are too. $\square$

The next lemma displays some well-known properties of Stirling numbers of the second kind.

Lemma 2.2. *Let K and r be nonnegative integers. Put*

$$S(r,K) = \frac{1}{K!}\sum_{k=0}^{K}(-1)^{K-k}\binom{K}{k}k^r.$$

Then $S(r,K)=0$ if $r<K$, $S(r,K)=1$ if $r=K$ and $S(r,K)\in\mathbb{N}$ if $r>K>0$.

Proof. See [1, Section III.2]. $\square$

Lemma 2.3. *Let a, b, n, K and N be positive integers with $a>0$, $b\geqslant 0$, $K\geqslant 0$, $N>0$, $N+K+n>0$. Then*

$$\sum_{k=0}^{K}(-1)^k\binom{K}{k}\frac{(a+b)_{a,N+k-1}}{(a+b)_{a,N+k+n}} = \begin{cases} \frac{(K+n)!a^K}{n!(aN+b)_{a,K+n+1}} & \text{if } n\geqslant 0,\\ 0 & \text{if } n\in\{-1,-2,\ldots,-K\}.\end{cases} \tag{3}$$

Proof. For $n\geqslant 0$ we use induction on K. For $K=0$ the identity follows by direct calculation. Suppose the assertion is true for $K-1$. Then, by the induction hypothesis,

$$\begin{aligned}
&\sum_{k=0}^{K}(-1)^k\binom{K}{k}\frac{(a+b)_{a,N+k-1}}{(a+b)_{a,N+k+n}}\\
&\quad=\sum_{k=0}^{K-1}(-1)^k\binom{K-1}{k}\frac{(a+b)_{a,N+k-1}}{(a+b)_{a,N+k+n}} - \sum_{k=0}^{K-1}(-1)^k\binom{K-1}{k}\frac{(a+b)_{a,N+k}}{(a+b)_{a,N+k+n+1}}\\
&\quad=\frac{(K-1+n)!a^{K-1}}{n!(aN+b)_{a,K+n}} - \frac{(K-1+n)!a^{K-1}}{n!(a(N+1)+b)_{a,K+n}} = \frac{(K+n)!a^K}{n!(aN+b)_{a,K+n+1}}.
\end{aligned}$$

For $n\in\{-1,-2,\ldots,-K\}$ we observe that

$$\frac{(a+b)_{a,N+k-1}}{(a+b)_{a,N+k+n}} = \prod_{j=n+1}^{-1}\bigl(a(N+k+j)+b\bigr)\in\mathbb{Z}[k]$$

of degree $-n-1<K$. Hence it can be written as $\sum_{r=0}^{K-1}a_r k^r$ and we apply Lemma 2.2 to $(-1)^K K!\sum_{r=0}^{K-1}a_r S(r,K)$. $\square$

Lemma 2.3 is essentially a result on the K-th iterate of a difference operator. This is clear from Corollary 2.1 the left-hand side of which represents

$$(-1)^K\Delta^K\frac{1}{N(N+1)\cdots(N+n)},$$

where $\Delta f(x) = f(x+1)-f(x)$. By choosing $a=1$, $b=0$ in Lemma 2.3, we get:

Corollary 2.1. *Let n, K and N be positive integers with $K \geqslant 0$, $N > 0$, $N+K+n > 0$. Then*

$$\sum_{k=0}^{K}(-1)^k\binom{K}{k}\frac{(N+k-1)!}{(N+k+n)!} = \begin{cases} \frac{(K+n)!(N-1)!}{n!(N+K+n)!} & \text{if } n \geqslant 0, \\ 0 & \text{if } n \in \{-1,-2,\dots,-K\}. \end{cases} \tag{4}$$

3. Basic theorems

We prove some basic results. Those based on Proposition 3.1 are more useful if the geometric sequence has length $< 4N$, the ones based on Proposition 3.2 if the length is greater.

Proposition 3.1. *Let a and b be integers with $a > 0$, $b \geqslant 0$. Let $(a_n)_{n=1}^{\infty}$ be a sequence of integers such that for infinitely many N the sequence*

$$a_{N-K}, a_{N-K+1}, \dots, a_{N+K+H}$$

forms a nonzero geometric sequence with quotient c/d where $c = c(N)$ and $d = d(N)$ are positive coprime integers and where $K = K(N)$ and $H = H(N)$ with $N - K \to \infty$ as $N \to \infty$. Set

$$A_K = \begin{cases} \gcd(K!, a^K) & \text{if } b \neq 0, \\ 1 & \text{otherwise}. \end{cases}$$

Assume that

$$\sum_{k=0}^{K}(-1)^k\binom{K}{k}c^{K-k}d^k\sum_{n=H}^{\infty}\frac{a_{N+n+k}}{(a(N+k)+b)_{a,n+1}} = o\left(\frac{a_N a^K c^K K!}{(aN+b)_{a,K+1}}\right) \tag{5}$$

and

$$a_N a^K c^K\sum_{n=0}^{H-1}\left(\frac{c}{d}\right)^n\frac{(K+n)!}{n!(aN+b)_{a,K+n+1}} = o\left(\frac{K!}{A_K}\right). \tag{6}$$

Then

$$S = \sum_{n=1}^{\infty}\frac{a_n}{(a+b)_{a,n}} \notin \mathbb{Q}.$$

Proof. Suppose $S = \frac{t}{q}$ where $t, q \in \mathbb{Z}$ and $q > 0$. Let N be a sufficiently large positive integer such that $a_{N-K}, a_{N-K+1}, \dots, a_{N+K+H}$ is a geometric sequence with quotient $\frac{c}{d}$ where $c = c(N)$ and $d = d(N)$ are positive coprime integers. Defining S_N as in (1) we have $qS_N \in \mathbb{Z}$ by Lemma 2.1. Hence, for $N > 0$,

$$D_N := q\sum_{k=0}^{K}(-1)^k\binom{K}{k}c^{K-k}d^k S_{N+k} \in \mathbb{Z}. \tag{7}$$

From this, the fact that $a_N, a_{N+1}, \dots, a_{N+K+H}$ form a geometric sequence with quotient $\frac{c}{d}$, Lemma 2.3 and (5), we obtain

$$\begin{aligned}D_N &= qa_N c^K \sum_{n=0}^{H-1}\left(\frac{c}{d}\right)^n \sum_{k=0}^{K}(-1)^k\binom{K}{k}\frac{(a+b)_{a,N+k-1}}{(a+b)_{a,N+k+n}} \\ &\quad + q\sum_{k=0}^{K}(-1)^k\binom{K}{k}c^{K-k}d^k\sum_{n=H}^{\infty}\frac{a_{N+n+k}}{(a(N+k)+b)_{a,n+1}} \\ &= qa_N a^K c^K \sum_{n=0}^{H-1}\left(\frac{c}{d}\right)^n\frac{(K+n)!}{n!(aN+b)_{a,K+n+1}} + o\left(\frac{a_N a^K c^K K!}{(aN+b)_{a,K+1}}\right).\end{aligned}$$

This and (6) imply

$$0 < \frac{|a_N| a^K c^K K!}{2(aN+b)_{a,K+1}} < |D_N| = o\left(\frac{K!}{A_K}\right). \tag{8}$$

Formulas (7) and (2) also yield

$$D_N = \sum_{k=0}^{K}(-1)^k\binom{K}{k}c^{K-k}d^k(a+b)_{a,N+k-1}\left(t - q\sum_{n=1}^{N+k-1}\frac{a_n}{(a+b)_{a,n}}\right).$$

Because of the geometric sequence property, we have

$$\begin{aligned}&\sum_{k=0}^{K}(-1)^k\binom{K}{k}c^{K-k}d^k(a+b)_{a,N+k-1}\sum_{n=N-K+k}^{N+k-1}\frac{a_n}{(a+b)_{a,n}} \\ &\quad = \sum_{k=0}^{K}(-1)^k\binom{K}{k}c^{K-k}d^k\sum_{n=0}^{K-1}a_{N-K+k+n}\frac{(a+b)_{a,N+k-1}}{(a+b)_{a,N-K+k+n}} \\ &\quad = \sum_{n=0}^{K-1}c^K a_{N-K+n}\sum_{k=0}^{K}(-1)^k\binom{K}{k}\frac{(a+b)_{a,N+k-1}}{(a+b)_{a,N+k+n-K}} = 0,\end{aligned}$$

the last identity by Lemma 2.3. We obtain

$$D_N = \sum_{k=0}^{K}(-1)^k\binom{K}{k}c^{K-k}d^k\left(t(a+b)_{a,N+k-1} - q\sum_{n=1}^{N-K+k-1}a_n\frac{(a+b)_{a,N+k-1}}{(a+b)_{a,n}}\right).$$

It follows that the integer D_N is divisible by $K!/A_K$. This contradicts (8). □

Remark 3.1. The only place where the condition $c/d > 0$ is used is in (8). Suppose $c/d < 0$. If

$$-\frac{c}{d}\cdot\frac{K+1}{a(N+K+1)+b} < \frac{1}{2},$$

then we find, by using that the terms are alternating and are more than halving in absolute value each time,

$$|a_N| a^K c^K \sum_{n=0}^{H-1}\left(\frac{c}{d}\right)^n\frac{(K+n)!}{n!(aN+b)_{a,K+n+1}} > \frac{1}{4}|a_N| a^K c^K\frac{K!}{(aN+b)_{a,K+1}}$$

and we find again that $S \notin \mathbb{Q}$.

It is possible to derive results similar to Proposition 3.1 and Theorem 3.1 in case $(a_n)_{n=1}^{\infty}$ is a sequence of Gaussian integers containing long arithmetic progressions, cf. Proposition 3.2. In order to derive the nonzeroness in (8) one has to impose an inequality as above, but with a slightly smaller right-hand side than 1/2.

We denote by $\lceil x\rceil$ and $\lfloor x\rfloor$ the smallest integer at least x and the largest integer at most x, respectively.

Theorem 3.1. *Let $0<\delta<1$. Let $(R(n))_{n=1}^{\infty}$ be a sequence of positive integers. Let $(a_n)_{n=1}^{\infty}$ be a sequence of integers such that for infinitely many $N\in\mathbb{N}$ the sequence $a_{N-2R(N)}, a_{N-2R(N)+1}, \dots, a_{\lceil N+5R(N)/(1-\delta)\rceil}$ forms a geometric progression and*

$$a_{N+n}=o\big(N^{R(N)+\delta n}\big)\quad \text{for } n=0,1,\dots\quad \text{and}\quad N-2R(N)\to\infty\quad \text{as } N\to\infty.$$

Then

$$S=\sum_{n=1}^{\infty}\frac{a_n}{n!}\notin\mathbb{Q}.$$

Proof. Let $a_{N+1}/a_N=c/d$ where $c=c(N)$ and $d=d(N)$ are coprime positive integers. We apply Proposition 3.1 with $a=1$, $b=0$, $K=2R$, $H=\lceil\frac{3+2\delta}{1-\delta}R\rceil$ with $R=R(N)$. It suffices to verify (5) and (6) as the rest is obvious. We observe that $c^{K-k}a_{N+k-K}=d^{K-k}a_N$ and $d^k a_{N+k}=c^k a_N$ for $0\leqslant k\leqslant K$ so that $c^{K-k}d^k|a_N$ and

$$\max_{k=0,\dots,K}c^{K-k}d^k\leqslant|a_N|=o\big(N^R\big).$$

Hence

$$\begin{aligned}\left|\sum_{k=0}^{K}(-1)^k\binom{K}{k}c^{K-k}d^k\sum_{n=H}^{\infty}\frac{a_{N+n+k}}{\frac{(N+n+k)!}{(N+k-1)!}}\right|&=o\left(|a_N|\sum_{k=0}^{K}\binom{K}{k}N^R\sum_{n=H}^{\infty}\frac{N^{\delta(n+k)}}{N^{n+1}}\right)\\&=o\left(|a_N|N^{R-1}\sum_{k=0}^{K}\binom{K}{k}N^{\delta k}\sum_{n=H}^{\infty}N^{-(1-\delta)n}\right)\\&=o\big(|a_N|2^KN^{R-1+\delta K-(1-\delta)H}\big)\\&=o\big(|a_N|2^KN^{-2R-1}\big)\\&=o\left(\frac{a_NK!(N-1)!}{(N+K)!}\right).\end{aligned}$$

Thus (5) is satisfied. Furthermore,

$$\begin{aligned}\left|a_Nc^K\sum_{n=0}^{H-1}\left(\frac{c}{d}\right)^j\frac{(K+n)!(N-1)!}{n!(N+K+n)!}\right|&=o\left(|a_N|N^R\frac{K!(N-1)!}{(N+K)!}\sum_{n=0}^{H-1}\frac{K^n}{N^n}\right)\\&=o\left(N^{2R}\frac{K!}{N^{K+1}}H\right)=o(K!).\end{aligned}$$

Since $A_K=1$ we find that (6) is also satisfied. □

By choosing $\delta = 1/6$ we obtain the following result.

Corollary 3.1. *Let $(R(n))_{n=1}^{\infty}$ be a sequence of positive integers. Let $(a_n)_{n=1}^{\infty}$ be a sequence of integers such that for infinitely many $N \in \mathbb{N}$ the sequence $a_{N-2R(N)}, a_{N-2R(N)+1}, \ldots, a_{N+6R(N)}$ forms a geometric progression and*

$$a_{N+n} = o\bigl(N^{R(N)+n/6}\bigr) \quad \text{for } n = 0, 1, \ldots \quad \text{and} \quad N - 2R(N) \to \infty \quad \text{as } N \to \infty.$$

Then

$$S = \sum_{n=1}^{\infty} \frac{a_n}{n!} \notin \mathbb{Q}.$$

Although the results from the present paper are not very suitable to derive linear independence results, it enables us to treat some case we could not handle before. We give an example.

Corollary 3.2. *The numbers*

$$\alpha_m = \sum_{n=1}^{\infty} \frac{(\pi(n))^m}{n!}$$

where $m \in \{0, 1, 2, \ldots\}$ and the number 1 are linearly independent over $\mathbb{Q}$.

Proof. Suppose that there exist $T \in \mathbb{N}$, $m_1, m_2, \ldots, m_T \in \{0, 1, 2, \ldots\}$ with $m_1 < m_2 < \cdots < m_T$ and $A_1, A_2, \ldots, A_T \in \mathbb{Z} \setminus \{0\}$ such that $S := \sum_{r=1}^{T} A_r \alpha_{m_r} \in \mathbb{Z}$. Then we can write

$$S = \sum_{r=1}^{T} A_r \alpha_{m_r} = \sum_{r=1}^{T} A_r \sum_{n=1}^{\infty} \frac{(\pi(n))^{m_r}}{n!} = \sum_{n=1}^{\infty} \frac{\sum_{r=1}^{T} A_r (\pi(n))^{m_r}}{n!}.$$

There exist infinitely many N such that $\pi(N - 2\lfloor \log N \rfloor) = \cdots = \pi(N + 6\lceil \log N \rceil)$, hence $\sum_{r=1}^{T} A_r (\pi(n))^{m_r}$ is constant for $n = N - 2\lfloor \log N \rfloor, \ldots, N + 6\lceil \log N \rceil$. Apply Corollary 3.1 with $R(N) = [\log N]$. We conclude that the number S is irrational and therefore the numbers 1, $\alpha_0, \alpha_1, \ldots$ are linearly independent over $\mathbb{Q}$. □

Open problem 3.1. *Prove the irrationality of*

$$\sum_{n=1}^{\infty} \frac{(\pi(n))^n}{n!}.$$

Proposition 3.2. *Let a and b be fixed integers with $a > 0$, $b \geqslant 0$. Let $(a_n)_{n=1}^{\infty}$ be a sequence of Gaussian integers such that $a_N, a_{N+1}, \ldots, a_{4N}$ form a geometric sequence for infinitely many N. Assume that*

$$a_n = o\bigl(n^{n/7}\bigr) \tag{9}$$

for n sufficiently large. Then

$$S \notin \mathbb{Q}[i] \quad \text{or} \quad a^N \sum_{n=0}^{N-1} a_{2N+n} \frac{(N+n)!}{n!(2aN+b)_{a,N+n+1}} = o\bigl(N^{-N/8}\bigr)$$

for infinitely many such N.

Proof. Suppose that $S = \frac{t}{q}$ where $t \in \mathbb{G}$, $q \in \mathbb{N}$. Let N be sufficiently large positive integer such that

$$a_N, a_{N+1}, \ldots, a_{4N}$$

is a geometric sequence with quotient $\frac{c}{d}$ where $c \in \mathbb{G}$ and $d \in \mathbb{N}$ are coprime. Then, by Lemma 2.1, $qS_N \in \mathbb{G}$ for all integers N. From this we obtain that

$$D_N := q \sum_{k=0}^{N} (-1)^k \binom{N}{k} c^{N-k} d^k S_{2N+k} \in \mathbb{G}. \tag{10}$$

This implies

$$D_N = q \sum_{k=0}^{N} (-1)^k \binom{N}{k} c^{N-k} d^k \sum_{n=0}^{\infty} \frac{a_{2N+k+n}}{(a(2N+k)+b)_{a,n+1}}.$$

Since $a_N, a_{N+1}, \ldots, a_{4N}$ is a geometric sequence with quotient c/d we have $c^N | a_{2N}$ and $d^N | a_{2N}$. Hence $|c^{N-k} d^k| \leqslant |a_{2N}| \leqslant (2N)^{2N/7}$ for $0 \leqslant k \leqslant N$. It follows from (9) that

$$\begin{aligned}
&\left| \sum_{k=0}^{N} (-1)^k \binom{N}{k} c^{N-k} d^k \left(\sum_{n=N}^{\infty} \frac{a_{2N+k+n}}{(a(2N+k)+b)_{a,n+1}} \right) \right| \\
&\quad \leqslant 2^N (2N)^{2N/7} \sum_{n=N+1}^{\infty} \frac{(3N+n)^{(3N+n)/7}}{(n+1)!} \\
&\quad \leqslant 2^{9N/7} N^{2N/7} \frac{(4N)^{4N/7} e^N}{N^N} \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots \right) = o\big(N^{-N/8}\big).
\end{aligned}$$

On the other hand, by the geometric progression condition and Lemma 2.3,

$$\begin{aligned}
&\sum_{k=0}^{N} (-1)^k \binom{N}{k} c^{N-k} d^k \sum_{n=0}^{N-1} \frac{a_{2N+k+n}}{(a(2N+k)+b)_{a,n+1}} \\
&\quad = c^N \sum_{n=0}^{N-1} a_{2N+n} \sum_{k=0}^{N} (-1)^k \binom{N}{k} \frac{(a+b)_{a,2N+k-1}}{(a+b)_{a,2N+k+n}} \\
&\quad = c^N \sum_{n=0}^{N-1} a_{2N+n} \frac{a^N (N+n)!}{n!(2aN+b)_{a,N+n+1}}.
\end{aligned}$$

Thus

$$D_N = q a^N c^N \sum_{n=0}^{N-1} a_{2N+n} \frac{(N+n)!}{n!(2aN+b)_{a,N+n+1}} + o\big(N^{-N/8}\big). \tag{11}$$

Hence

$$|D_N| < \left|qa^N c^N \sum_{n=0}^{N-1} (2N+n)^{(2N+n)/7} \frac{(N+n)!}{n!(N+n)!}\right| + o\big(N^{-N/8}\big)$$
$$< \left|eqa^N c^N (3N)^{3N/7}\right| + o\big(N^{-N/8}\big) < (N!)^{6/7} \tag{12}$$

for N sufficiently large.

Eqs. (10) and (2) yield, by Lemma 2.1,

$$D_N = q\sum_{k=0}^{N}(-1)^k\binom{N}{k}c^{N-k}d^k S_{2N+k}$$
$$= \sum_{k=0}^{N}(-1)^k\binom{N}{k}c^{N-k}d^k(a+b)_{a,2N+k-1}\left(t - q\sum_{n=1}^{2N+k-1}\frac{a_n}{(a+b)_{a,n}}\right).$$

We have, by Lemma 2.3,

$$\sum_{k=0}^{N}(-1)^k\binom{N}{k}c^{N-k}d^k(a+b)_{a,2N+k-1}\left(\sum_{n=N+k}^{2N+k-1}\frac{a_n}{(a+b)_{a,n}}\right)$$
$$= \sum_{k=0}^{N}(-1)^k\binom{N}{k}c^{N}(a+b)_{a,2N+k-1}\left(\sum_{n=0}^{N-1}\frac{a_{N+n}}{(a+b)_{a,N+k+n}}\right)$$
$$= \sum_{n=0}^{N-1}a_{N+n}c^N\sum_{k=0}^{N}(-1)^k\binom{N}{k}\frac{(a+b)_{a,2N+k-1}}{(a+b)_{a,2N+k+(n-N)}} = 0.$$

Hence

$$D_N = \sum_{k=0}^{N}(-1)^k\binom{N}{k}c^{N-k}d^k\left(t(a+b)_{a,2N+k-1} - q\sum_{n=1}^{N+k-1}\frac{a_n(a+b)_{a,2N+k-1}}{(a+b)_{a,n}}\right). \tag{13}$$

This is an integer divisible by $N!/A_N$. Observe that

$$A_N = \prod_{p|a} p^{[\frac{N}{p}]+[\frac{N}{p^2}]+\cdots} < \prod_{p|a} p^{N/(p-1)} < C^N,$$

for some fixed number C. Hence $N!/A_N > (N!)^{6/7}$ which combined with (12) implies that $D_N = 0$. □

Theorem 3.2. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive integers such that $a_N, a_{N+1}, \ldots, a_{4N}$ form a geometric sequence for infinitely many N. Assume that*

$$a_n = o\big(n^{n/7}\big) \tag{14}$$

for n sufficiently large. Then

$$S = \sum_{n=1}^{\infty} \frac{a_n}{n!} \notin \mathbb{Q}.$$

Proof. According to the proof of Proposition 3.2 it suffices to show that $D_N \neq 0$. All the terms are positive. Hence, by (11) and Stirlings' formula,

$$|D_N| > qa^N c^N a_{2N} \frac{N!(2N-1)!}{(3N)!} + o\big(N^{-N/8}\big) > \frac{1}{7^N}$$

for all sufficiently large N. □

Theorem 3.3. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of Gaussian integers such that for infinitely many N the terms $a_N, a_{N+1}, \ldots, a_{4N}$ form a geometric sequence with $|a_{N+1}/a_N| \leqslant 2$. Assume that*

$$a_n = o\big(n^{n/7}\big) \tag{15}$$

for n sufficiently large. Then

$$S = \sum_{n=1}^{\infty} \frac{a_n}{n!} \notin \mathbb{Q}[i].$$

Proof. In view of the proof of Proposition 3.2 it suffices to show that $D_N \neq 0$. We have, by (11) with $a = 1$, $b = 0$,

$$D_N = qc^N a_{2N} \sum_{n=0}^{N-1} \left(\frac{c}{d}\right)^n \frac{(N+n)!(2N-1)!}{n!(3N+n)!} + o\big(N^{-N/8}\big).$$

Hence D_N equals

$$qc^N a_{2N} \frac{(N)!(2N-1)!}{(3N)!} \left(1 + \frac{c(N+1)}{d(3N+1)} + \frac{c^2(N+1)(N+2)}{2!d^2(3N+1)(3N+2)} + \cdots\right) + o\big(N^{-N/8}\big).$$

By $|c/d| = |a_{N+1}/a_N| \leqslant 2$ it follows that the expression between big parentheses minus 1 is smaller than

$$\big(1+o(1)\big)\left(\frac{2}{3} + \frac{2^2}{2!(3^2)} + \frac{2^3}{3!(3^3)} + \frac{2^4}{4!(3^4)} + \cdots\right) = \big(1+o(1)\big)\big(e^{2/3}-1\big).$$

Since $e^{2/3} - 1 < 0.95$, the omitted term 1 is asymptotically larger and we obtain

$$|D_N| > \frac{1}{20} qc^N |a_{2N}| \frac{N!(2N-1)!}{(3N)!} > \frac{1}{20} \cdot \frac{1}{7^N}$$

for N sufficiently large. Hence $D_N \neq 0$. □

4. On the irrationality of sums $\sum m^{b_n}/n!$

Theorem 4.1. *Let m be a fixed integer with $m > 1$. Let $(b_n)_{n=1}^{\infty}$ be a sequence of positive integers such that*

$$\liminf_{n\to\infty}\frac{b_n}{n} < \frac{1}{m-1} \tag{16}$$

and

$$m^{b_n - b_{n-1}} < \frac{n}{2} \tag{17}$$

for all large n. Then

$$\sum_{n=1}^{\infty}\frac{m^{b_n}}{n!} \notin \mathbb{Q}.$$

Proof. Suppose $\sum_{n=1}^{\infty}\frac{m^{b_n}}{n!} = \frac{t}{q}$ with $t, q \in \mathbb{Z}$, $q > 0$, $(t,q) = 1$. Then, according to Lemma 2.1,

$$0 < \sum_{n=N+1}^{\infty}\frac{m^{b_n}}{(N+1)\cdots n} = \frac{tN!}{q} - \sum_{n=1}^{N}\frac{N!}{n!}m^{b_n}.$$

For $N \geqslant q$, the right-hand side is an integer divisible by

$$m^{[\frac{N}{m}]+[\frac{N}{m^2}]+\cdots+\min(-\operatorname{ord}_m(q),\min_{n\leqslant N}(b_n-[\frac{n}{m}]-[\frac{n}{m^2}]-\cdots))}.$$

On the other hand, by (17) we have, for sufficiently large N,

$$m^{b_N}\left(\frac{m^{b_{N+1}-b_N}}{N+1} + \frac{m^{b_{N+1}-b_N}}{N+1}\frac{m^{b_{N+2}-b_{N+1}}}{N+2} + \cdots\right) < m^{b_N}.$$

Thus, for N sufficiently large, we obtain

$$\left[\frac{N}{m}\right] + \left[\frac{N}{m^2}\right] + \cdots - \max\left(\operatorname{ord}_m(q), \max_{n\leqslant N}\left(-b_n + \left[\frac{n}{m}\right] + \left[\frac{n}{m^2}\right] + \cdots\right)\right) < b_N. \tag{18}$$

From (16) we see that

$$\limsup\left(-b_n + \left[\frac{n}{m}\right] + \left[\frac{n}{m^2}\right] + \cdots\right) = \infty.$$

Hence arbitrarily large values of N exist so that

$$-b_N + \left[\frac{N}{m}\right] + \left[\frac{N}{m^2}\right] + \cdots \geqslant \max_{n\leqslant N}\left(-b_n + \left[\frac{n}{m}\right] + \left[\frac{n}{m^2}\right] + \cdots\right),$$

and that the left-hand side exceeds q. This gives a contradiction with (18). □

The following consequence shows in particular that $\sum_{n=1}^{\infty}\frac{m^{\pi(n)}}{n!}$ is irrational for $m = 0, 1, 2, \ldots$.

Corollary 4.1. *Let m be a positive integer. Let A be an infinite subset of $\mathbb{N}$ with lower asymptotic density less than $1/(m-1)$. Denote by b_n the number of elements at most n in A. Then*

$$S = \sum_{n=1}^{\infty} \frac{m^{b_n}}{n!} \notin \mathbb{Q}.$$

Proof. The conditions (16) and (17) are satisfied. □

Theorems 3.1 and 3.2 can directly be applied to sequences $\sum_{n=1}^{\infty} \frac{m^{b_n}}{n!}$. See for example the following result.

Theorem 4.2. *Let m be a positive integer. Let $(b_n)_{n=1}^{\infty}$ be a sequence of positive integers such that $b_N, b_{N+1}, \dots, b_{4N}$ form an arithmetic progression for infinitely many positive integers N. Suppose that*

$$b_n = o(n \log n).$$

Then the number

$$\sum_{n=1}^{\infty} \frac{m^{b_n}}{n!}$$

is irrational.

Proof. Apply Theorem 3.2. □

By choosing $b_n = n$ in Theorem 4.2 we immediately obtain that

$$e^m = \sum_{n=0}^{\infty} \frac{m^n}{n!} \notin \mathbb{Q}.$$

The irrationality of π does not follow immediately from Proposition 3.2, but can be deduced from its proof by a simple trick.

Theorem 4.3. *Let m be a Gaussian integer. Let $(b_n)_{n=1}^{\infty}$ be a sequence of positive integers for which $b_N, b_{N+1}, \dots, b_{4N}$ form an arithmetic progression for infinitely many positive integers N such that $2N-1$ is prime. Assume that $b_n = o(n \log n)$. Then the number*

$$\sum_{n=1}^{\infty} \frac{m^{b_n}}{n!} \notin \mathbb{Q}[i].$$

Proof. Put $m^{b_{N+1}/b_N} = c/d$ where c is a Gaussian integer and d a positive integer which is coprime to c. According to the proof of Proposition 3.2 it suffices to prove that $D_N \neq 0$. We have by (13) with $a = 1$, $b = 0$ that

$$D_N = \sum_{k=0}^{N} (-1)^k \binom{N}{k} c^{N-k} d^k \left(t(2N+k-1)! - q \sum_{n=1}^{N+k-1} m^{b_n} \frac{(2N+k-1)!}{n!} \right).$$

This implies that D_N is a Gaussian integer divisible by $N!$. Moreover, since $2N-1$ is a prime, all terms are divisible by $2N-1$ except for the one with $k = N$, $n = 2N-1$. Thus D_N is not divisible by $2N-1$ and therefore nonzero. □

Corollary 4.2. π *is irrational.*

Proof. Suppose that $\pi = \frac{t}{q}$ where $t, q \in \mathbb{N}$. Then we can write

$$(-1)^q = e^{iq\pi} = e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}.$$

However, the sum on the right-hand side is irrational by Theorem 4.3 applied with $m = it$ and $b_n = n$. $\square$

There are no big difficulties if the denominator is not a factorial, but a similar product of terms of an arithmetic progression. For example, the following result can be proved by adjusting the proof of Theorem 4.1.

Example 4.1. For $m = 1, 2, \ldots$ we have

$$\sum_{n=1}^{\infty} \frac{m^{\pi(n)}}{\prod_{j=1}^{n}(2j+1)} \notin \mathbb{Q}.$$

References

[1] M. Aigner, Combinatorial Theory, Classics Math., Springer, 1997.
[2] P. Erdős, Sur certaines series à valeur irrationnelle, Enseign. Math. 4 (1958) 93–100.
[3] P. Erdős, M. Kac, Problem 4518, Amer. Math. Monthly 61 (1954) 264.
[4] P. Erdős, E.G. Straus, On the irrationality of certain series, Pacific J. Math. 55 (1974) 85–92.
[5] N.I. Fel'dman, Yu.V. Nesterenko, Transcendental numbers, in: A.N. Parshin, I.R. Shafarevich (Eds.), Number Theory, IV, in: Encyclopaedia Math. Sci., vol. 44, Springer, Berlin, 1998.
[6] J.B. Friedlander, F. Luca, M. Stoiciu, On the irrationality of a divisor function series, Integers 7 (2007), A31, 9 pp. (electronic).
[7] J. Hančl, R. Tijdeman, On the irrationality of Cantor series, J. Reine Angew. Math. 571 (2004) 145–158.
[8] J. Hančl, R. Tijdeman, On the irrationality of factorial series, Acta Arith. 118 (2005) 383–401.
[9] J. Hančl, R. Tijdeman, On the irrationality of polynomial Cantor series, Acta Arith. 133 (2008) 37–52.
[10] J.F. Koksma, Diophantische Approximationen, Ergeb. Math. Grenzgeb., vol. 4, Springer, Berlin, 1936.
[11] Yu.V. Nesterenko, A simple proof of the irrationality of π, Russ. J. Math. Phys. 13 (2006) 473.
[12] I. Niven, Irrational Numbers, Carus Math. Monogr., vol. 11, Math. Ass. Amer., Wiley, 1963.
[13] J.-C. Schlage-Puchta, The irrationality of a number theoretical series, Ramanujan J. 12 (2006) 455–460.
[14] J.-C. Schlage-Puchta, The irrationality of some number theoretical series, Acta Arith. 126 (2007) 295–303.
[15] A.B. Shidlovskij, Transcendental Numbers, Walter de Gruyter, Berlin, 1989.
[16] R. Tijdeman, P.Z. Yuan, On the rationality of Cantor and Ahmes series, Indag. Math. (N.S.) 13 (2002) 407–418.